Reversible Lifted Implementation of Half-Sample Symmetric Filter Banks

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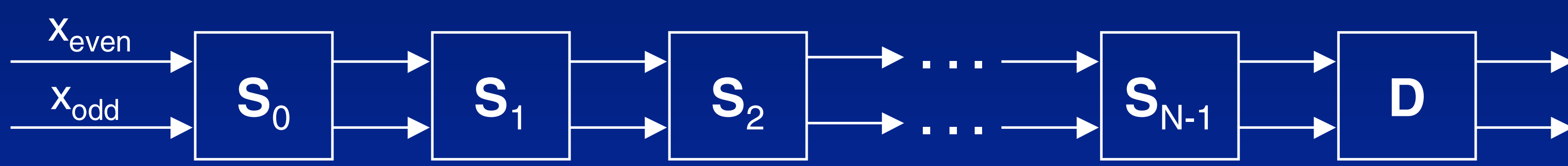
Overview

- Reversible lifting factorization of whole-sample symmetric (WS) filter banks (filter banks with odd-length impulse responses) is straightforward.
- Reversible factorization of half-sample symmetric (HS) filter banks (filter banks with even-length impulse responses) is more problematic. In particular, in contrast to the WS case, reversibility for HS filter banks is highly dependent on the particular choice of rounding rule(s) used to achieve reversibility.
- The issue of HS reversibility is intimately tied to the HS boundary-handling technique (i.e., (2,2)-symmetric extension) for HS filter banks.

Polyphase Lifting Factorization of Filter Banks

A polyphase factorization of an analysis filter bank has the following form:



In more compact 2x2 transfer matrix notation:

$$\mathbf{Q}(z) = \mathbf{D}\,\mathbf{S}_{N-1}(z)\ldots\mathbf{S}_1(z)\mathbf{S}_0(z)$$

D is a memoryless diagonal matrix that scales the output gains; **D** is generally omitted in reversible implementations. The $\mathbf{S}_i(z)$ are alternately lower or upper triangular, with one off-diagonal filter element; e.g.,

$$\mathbf{S}_i(z) = \begin{bmatrix} 1 & 0 \\ h_i(z) & 1 \end{bmatrix}$$

This step lifts the odd (highpass) channel by adding $x_{even}(z)h(z)$ to $x_{odd}(z)$, but it leaves x_{even} unchanged.

Lifting Structures for Linear Phase Filter Banks

WS (odd-length symmetric) filter banks use lifting steps $\mathbf{S}_i(z)$ whose off-diagonal filter element is HS (even-length symmetric). This is based on:

Theorem 1 If $\mathbf{Q}_0(z)$ is WS and $\mathbf{Q}(z) = \mathbf{S}(z)\mathbf{Q}_0(z)$ then $\mathbf{Q}(z)$ is WS if and only if the lifting step $\mathbf{S}(z)$ has an HS off-diagonal filter element.

The corresponding theorem for lifting HS filter banks is:

Theorem 2 If $\mathbf{Q}_0(z)$ is HS and $\mathbf{Q}(z) = \mathbf{S}(z)\mathbf{Q}_0(z)$ then $\mathbf{Q}(z)$ is HS if and only if $\mathbf{S}(z)$ has a WA (odd-length *antisymmetric*) off-diagonal filter element.

Problem Since the polyphase rep. starts out WS (the lazy wavelet), it's necessary to construct an HS "base filter bank" before applying Theorem 2:

$$\mathbf{Q}(z) = \mathbf{D}\ \underbrace{\mathbf{S}_{N-1}(z)\ldots\mathbf{S}_{i+1}(z)}_{|\leftarrow \text{WA steps} \rightarrow|}\ \underbrace{\mathbf{S}_i(z)\ldots.\mathbf{S}_0(z)}_{|\leftarrow \text{HS base} \rightarrow|}$$

If the partial product $\mathbf{S}_i(z)\ldots\mathbf{S}_0(z)$ generates an HS "base" filter bank (e.g., the Haar), then Theorem 2 says that higher-order HS filter banks can be lifted from it by applying WA lifting steps. *Figuring out how to construct an HS base is a significant problem (cf. WG1N1914, Canon Research France).*

Rounding Rules

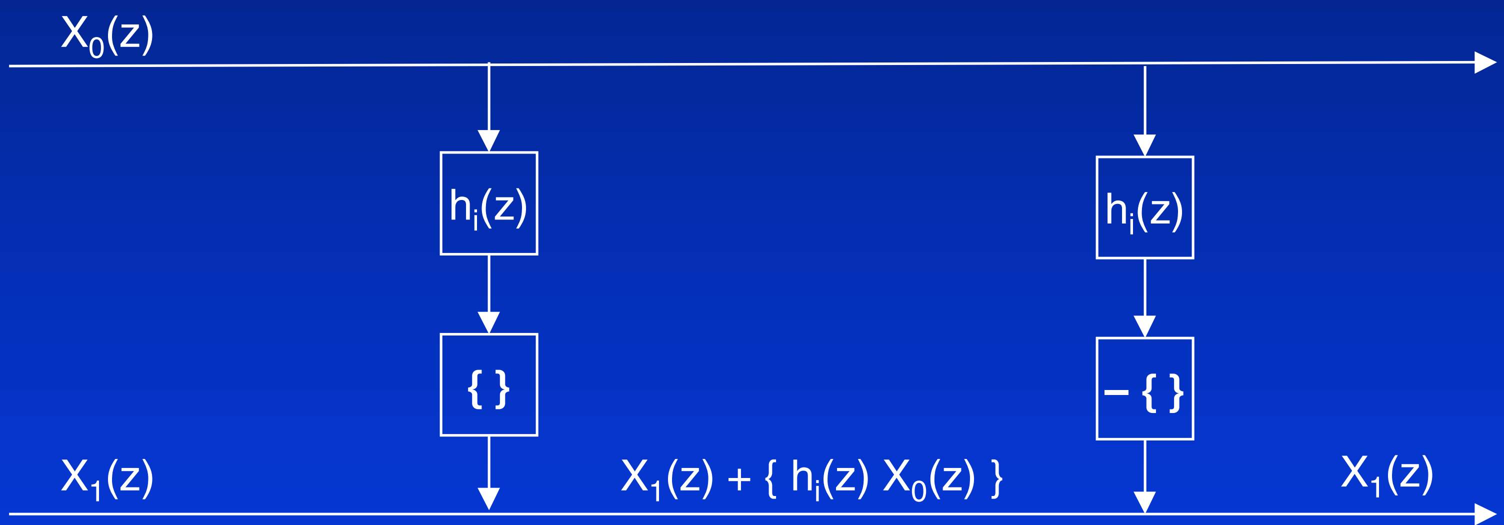
- In reversible implementations, rounding is used ensure that lifting updates are performed in integer arithmetic.

  Rounding rules are denoted generically by curly braces, {x}. Specific examples include:

$\lfloor x \rfloor$	floor function
$\lfloor x+\beta \rfloor$	floor with offset (e.g., β=1/2 is the current Annex G rule)
$\lceil x \rceil$	ceiling function
$[x]$	integer part (symmetric rounding)
$[x\pm\beta]$	integer part with symmetric offset

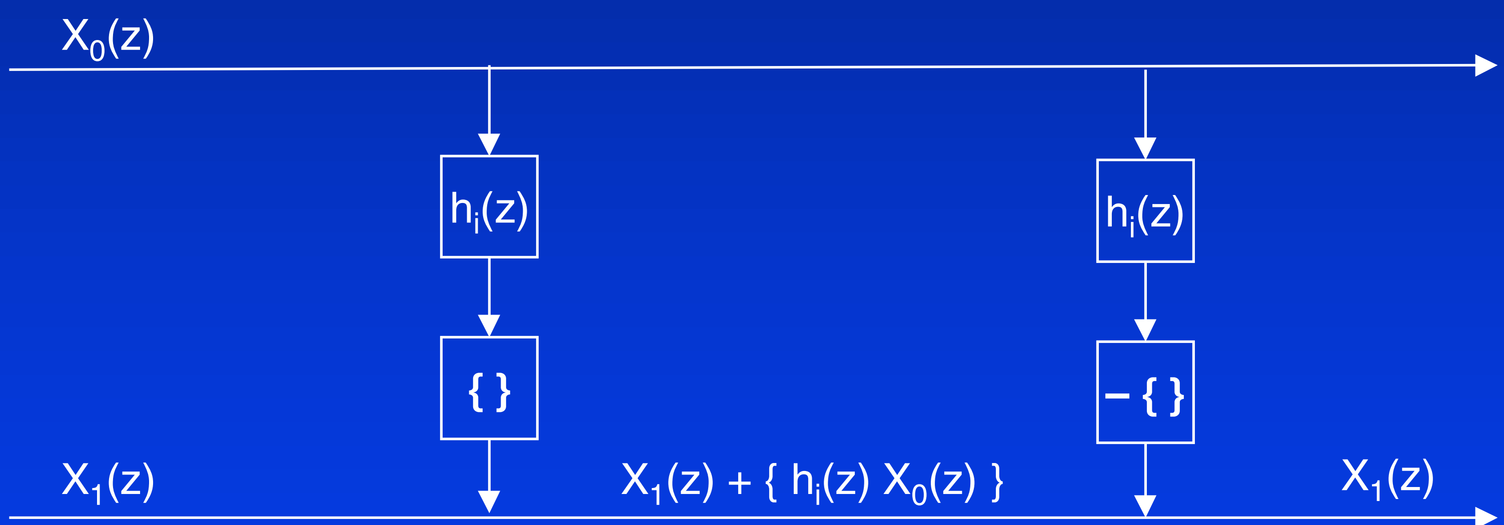
Reversible Lifting

In a reversible implementation, every lifting step filter element is followed by a rounding rule that's applied before the update. The block diagram for forward/inverse reversible odd-channel lifting steps is:



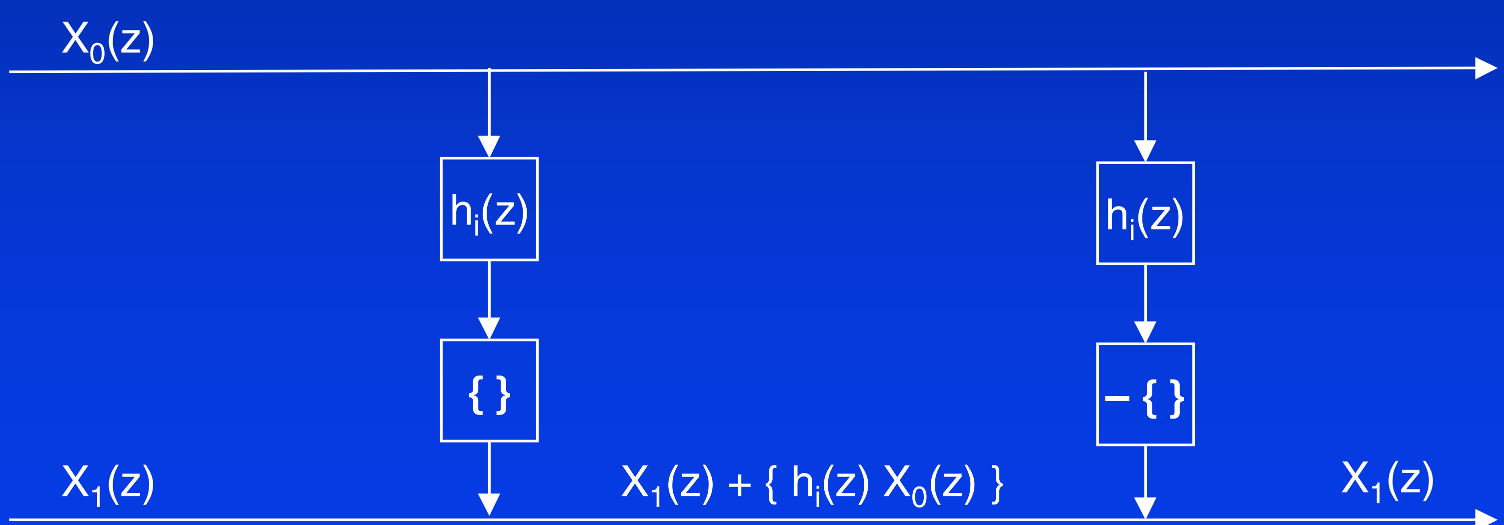
Preservation of Subband Symmetry

- **Assume both X_0 and X_1 are symmetric (WS filter banks) and $h_i(z)$ is an appropriate type of linear phase filter. If $h_i(z)X_0(z)$ has the same symmetry as $X_1(z)$ then so does the rounded value; therefore, the updated subband, $X_1(z) + \{ h_i(z)X_0(z) \}$, remains symmetric.**
- ***This means that symmetric extension in the synthesis bank will reproduce the correct boundary values for the truncated subband.***



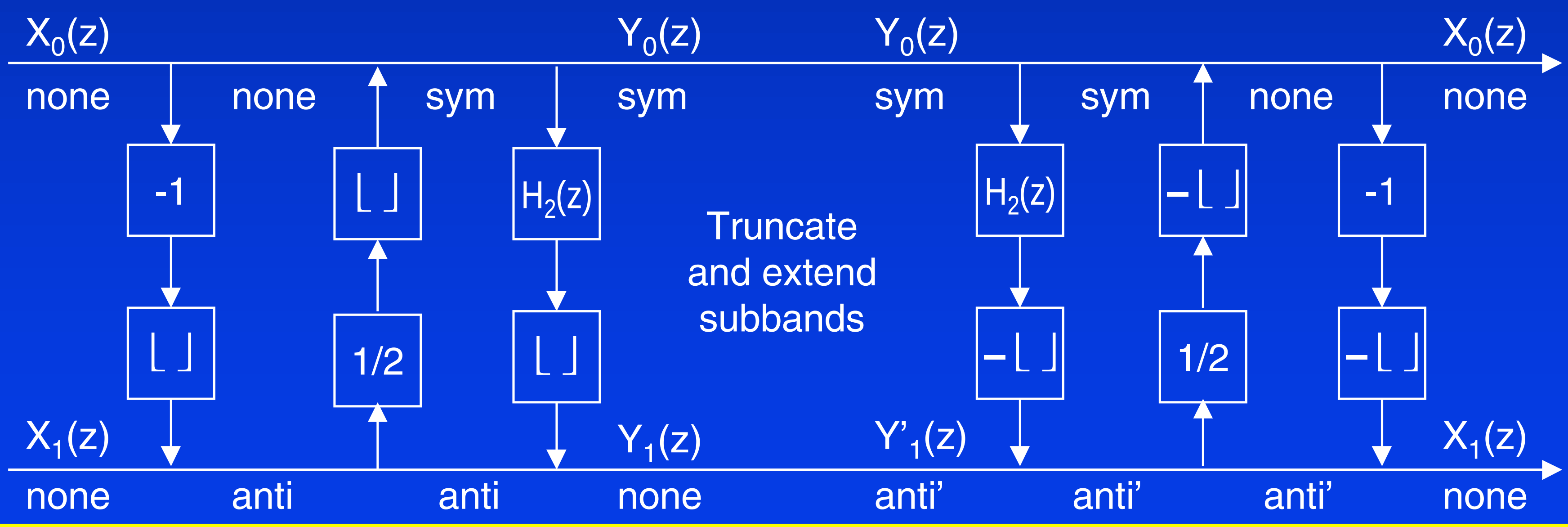
Preservation of Subband Antisymmetry

- **Now assume X_1 is antisymmetric (HS filter banks). If $h_i(z)X_0(z)$ has the same antisymmetry as $X_1(z)$ then the rounded value, and therefore the updated subband, $X_1(z) + \{ h_i(z)X_0(z) \}$, will be antisymmetric *if and only if* the rounding rule is an odd function: $\{-x\} = -\{x\}$.**
- ***If antisymmetry is broken by using an inappropriate rounding rule (e.g., the floor function) then truncated highpass subbands cannot be regenerated by symmetric extension in the synthesis bank! Moreover, subsequent lowpass lifting steps will destroy the symmetry of $X_0(z)$.***



Why the 2-6 and 2-10 Filter Banks are Reversible

- If we need to use an odd rounding rule such as [x] or, more generally, [x±β], to preserve linear phasse in HS subbands, why do the 2-6 and the 2-10 filter banks work with floor function rounding?
- These are lifted from Haar, which gives linear phase subbands starting from (nonsymmetric) demux'd input channels. Subband symmetry continues to hold if the Haar lifting steps are rounded using floor (but NOT using [ ] !), so $Y_0(z)$ is symmetric, and there are no more lowpass lifts. Thus, the steps lifted from Y_0 are successfully inverted, and the Haar steps invert because they don't feel the boundary of Y'_1.



Other Special Cases of HS Reversibility using Symmetric Extension Transforms

- All integer filter banks lifted from the Haar (not just 2-2N cases) can be implemented reversibly if the Haar base is rounded using floor (so that the rounded Haar base generates symmetric subbands) and the subsequent WA steps are rounded using symmetric truncation [x±β].
- In fact, any integer HS filter bank lifted from a *symmetry-generating* rounded HS base can be implemented reversibly provided the subsequent WA steps are rounded using an odd rounding function, such as symmetric truncation [x±β].
- This is tricky, kids: e.g., the HS base (transposed lifting of the Haar) for the 6-2 filter bank (defined by interchanging the 2-6 analysis and synthesis filters) is *not* symmetry-generating when rounded using either floor *or* symmetric truncation!
- Getting a symmetry-generating rounded HS base can be nontrivial; e.g., there exist elementary examples of HS filter banks generated using the CRF factorization (WG1N1914) that do not appear to admit reversible implementation in symmetric extension transforms using *any* known rounding rules.

Options

- Do nothing (i.e., continue using floor rounding, symmetric extension).
  - Pro: path of least resistance.
  - Con: severely limits reversible options---essentially the only reversible HS filter banks would be 2-2N filter banks.
- Signal rounding rules for each lifting step.
  - Pro: maximum flexibility.
  - Con: maximum overhead; not consistent with WS case; ugly.
- Signal distinction between steps generating HS base, which would use floor, and WA lifts, which would use symmetric truncation.
  - Pro: ensures reversibility for *all* filter banks lifted from Haar (not just 2-2N); allows signaling folded filters (less ATK marker segment overhead).
  - Con: some signaling & new definitions required; kludgy.
- Use interleaved extension (with floor rounding) between lifting steps.
  - Pro: appears to preserve reversibility in all cases.
  - Con: must be made equivalent to pre-extension in WS cases; significant change in normative text for Annex G.